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Near 1-Designs

W. G. BRIDGES

The City University of New York, New York, New York 10036

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A $n \times m$ (0, 1)-matrix with row and column sums at least two and column inner products one has $m \geqslant n$. The case $m = n$ is settled by a theorem of de Bruijn and Erdös [4]. This paper settles the case $m = n - 1$. It is found that such a matrix is a partial projective plane with one exception occurring for $n = 6$.

1. INTRODUCTION

In 1948 de Bruijn and Erdös [4] proved the following theorem:

Let $S_1, S_2, \ldots, S_n$ be subsets of the m-set $\{a_1, \ldots, a_m\}$. Suppose each a_i occurs in at least two of the sets S_j and that each S_i contains more than one element. If each distinct pair $\{a_i, a_j\}$ is contained in precisely one S_k then $m \leqslant n$. If $m = n$ the sets either form a projective plane or the configuration (modulo relabeling) is given by $\{a_2, \ldots, a_n\}, \{a_1, a_2\}, \ldots, \{a_1, a_n\}$.

This theorem has been generalized by Ryser [6] and Woodall [8] to the case in which each pair $\{a_i, a_j\}$ is contained in λ of the sets S_i. The inequality remains valid and the case $m = n$ is treated in detail [1–3, 7, 8]. The near extremal case $m = n - 1$ for general λ has been investigated, though not settled, under the additional assumption that each a_j occurs in the same number of sets S_i [1, 5, 8]. For $\lambda = 1$ these results say that, if $|S_i| \geqslant 2$, such a configuration is a projective plane with a point deleted. We show here that without restriction on the replications of the a_i the same result holds, with one exception, a configuration consisting of six subsets of a 5-set embedded in the finite plane of order 2.

For ease in stating the result we make the following definitions: Let $a_1, \ldots, a_{n-1}$ be distinct elements and let $\mathscr{S} = \{S_1, \ldots, S_n\}$ be a family of $n > 4$ distinct subsets of $\mathfrak{A} = \{a_1, \ldots, a_{n-1}\}$. Suppose each S_i contains at least two elements and that each a_j belongs to at least two members of $\mathscr{S}$. $\mathscr{S}$ is called a *near* 1-*design on* $\mathfrak{A}$ if each distinct pair $\{a_i, a_j\}$ is contained in precisely one element of $\mathscr{S}$. The near 1-design $\mathscr{S}$ is called a *partial plane* if there is a projective plane on the points $\{a_1, ..., a_{n-1}, a_n\}$, say $\{T_1, ..., T_n\}$, such that $S_i = T_i - \{a_n\}$. Throughout the number of elements of $\mathscr{S}$ containing a_j will be called the *replication number* of a_j and will be denoted k_j.

A near 1-design is represented in the obvious way by an $n \times (n-1)$ (0, 1)-matrix $A = (a_{ij})$ where $a_{ij} = 1$ if and only if $a_j \in S_i$. This matrix satisfies

$$A^t A = \operatorname{diag}(k_1 - 1, ..., k_{n-1} - 1) + J, \tag{1.1}$$

where J is the square matrix of order $n-1$ all of whose entries are one. The basic approach of this study is to use (1.1) to ascertain the structure of the dual design carried by A^t, that is, we determine AA^t. A fundamental tool will be the following matrix theory lemma appearing in [7].

LEMMA 1.1. *Let X and Y be real square matrices of order n such that*

$$XY = D + (\sqrt{\lambda_i \lambda_j}),$$

where $D = \operatorname{diag}(k_1 - \lambda_1, ..., k_n - \lambda_n)$ *and* $k_j > \lambda_j \geqslant 0$. *With* $\Delta = \det(XY)$ *and* $\pi = \pi_{j=1}^{n} (k_j - \lambda_j)$ *we have*

$$YD^{-1}X = I + \frac{\pi}{\Delta}(y_i x_j),$$

where

$$y_i = \sum_{j=1}^{n} \frac{\sqrt{\lambda_j}}{k_j - \lambda_j} y_{ij},$$

$$x_j = \sum_{i=1}^{n} \frac{\sqrt{\lambda_i}}{k_i - \lambda_i} x_{ij}.$$

The proof is quite direct and we refer the reader to [7].

2. THE THEOREM

THEOREM 2.1. *Let $\mathscr{S} = \{S_1, ..., S_n\}$ be a near 1-design on $\{a_1, ..., a_{n-1}\}$. Then $\mathscr{S}$ is a partial plane or $n = 6$ and the configuration may be relabeled as*

$$\{a_1, a_2, a_3\}, \{a_1, a_4, a_5\}, \{a_2, a_4\}, \{a_2, a_5\}, \{a_3, a_4\}, \{a_3, a_5\}. \tag{2.1}$$

The design (2.1) is represented by the following 6×5 (0,1)-matrix:

$$\begin{array}{ccccc} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array} \tag{2.2}$$

This matrix may be obtained from the incidence matrix of the plane of order two by selecting a row and deleting two columns with ones in this row and then deleting the selected row.

We recall that k_j will denote the j-th column sum of the representing matrix A of the near 1-design and we let S_i correspond to the i-th row of A. We further let r_i be the i-th row sum of A and throughout use the notation:

$$\begin{aligned} \pi &= \prod_{j=1}^{n-1} (k_j - 1), \\ \pi_i &= \pi/(k_i - 1), \qquad i = 1,\ldots, n-1, \\ \Delta &= \pi + \sum_{i=1}^{n-1} \pi_i , \\ D &= \operatorname{diag}(k_1 - 1,\ldots, k_{n-1} - 1). \end{aligned} \tag{2.3}$$

Observe that

$$\Delta = \det(D + J) = \det(A^t A) > 0. \tag{2.4}$$

Before proceeding to the proof of the theorem we note the equivalent matrix statement: *Let A be an $n \times n - 1$ $(0, 1)$-matrix with line sums at least two and column inner products one. Then A is the incidence matrix of a projective plane with a column removed or else A may be permuted to the matrix* (2.2).

3. Proof of Theorem 2.1

The proof of Theorem 2.1 will proceed through a series of lemmas. The first of these characterizes the design (2.1).

LEMMA 3.1. *Let A represent a near 1-design with some $k_j = 2$. Then to within row and column permutations A is the matrix* (2.2).

Proof. Take $k_1 = 2$ and write A in the normalized form

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ 0 & \mathbf{1} & & & & A_1 & \\ \cdot & & \mathbf{1} & & & \cdot & \\ \cdot & & & \cdot & & \cdot & \\ \cdot & & & & \cdot & \cdot & \\ 0 & & & \mathbf{1} & & A_{r_1-1} & \end{bmatrix}, \tag{3.1}$$

where $\mathbf{1}$ denotes a string of ones of the appropriate length. Note $r_i \geqslant 2$ prevents a row (after row 1) with zeros in the initial r_1 positions or the terminal r_2 positions. Evidently $r_1 + r_2 = n$ and the A_i are permutation matrices. This gives $k_2 = \cdots = k_{r_1} = r_2$ and similarly $k_{r_1+1} = \cdots = k_{n-1} = r_1$. Also $r_3 = \cdots = r_n = 2$ so that, from $\sum k_j = \sum r_i$, we deduce that $r_1 r_2 = 2n - 3$. This, with $r_1 + r_2 = n$, means that $n^2 - 4(2n - 3) = (n - 4)^2 - 4$ is an integral square, forcing $n = 6$ and the matrix (2.2).

In the following lemmas we attempt to determine the matrix AA^t. From (2.4) it follows that A has rank $n - 1$. Define a real vector $\boldsymbol{\eta} = (\eta_1, \dots, \eta_n)^t$ by

$$A^t \boldsymbol{\eta} = \mathbf{0}, \qquad \sum_{i=1}^{n} \eta_i^2 = 1. \tag{3.2}$$

Now adjoin $\boldsymbol{\eta}$ as a final column to A, obtaining the $n \times n$ matrix

$$C = [A\ \boldsymbol{\eta}], \tag{3.3}$$

satisfying

$$C^t C = \begin{bmatrix} D + J & 0 \\ & \vdots \\ & 0 \\ 0 \cdots 0 & 1 \end{bmatrix} = D_1 + (\sqrt{\lambda_i \lambda_j}), \tag{3.4}$$

where $D_1 = \operatorname{diag}(k_1 - 1, \dots, k_{n-1} - 1, 1)$ and $\lambda_1 = \cdots = \lambda_{n-1} = 1$, $\lambda_n = 0$. Now, by the Lemma 1.1 mentioned earlier, we have from (3.4)

$$C D_1^{-1} C^t = I + \frac{\pi}{\Delta} [\varphi_i \varphi_j], \tag{3.5}$$

where π and Δ are given in (2.3) and

$$\varphi_i = \sum_{j=1}^{n-1} \frac{a_{ij}}{k_j - 1}. \tag{3.6}$$

We may write this as

$$\varphi_{ij} + \eta_i\eta_j = \delta_{ij} + \frac{\pi}{\Delta}\varphi_i\varphi_j . \tag{3.7}$$

Set

$$\varphi_{ij} = \sum_{\ell=1}^{n-1} \frac{a_{i\ell}a_{j\ell}}{k_\ell - 1} \qquad (i, j = 1,..., n).$$

Since A is (0, 1) $\varphi_{ii} = \varphi_i$, and, for $i \neq j$, $\varphi_{ij} = 0$ if and only if $S_i \cap S_j = \varnothing$ while $\varphi_{ij} \neq 0$ is equivalent to $|S_i \cap S_j| = 1$. Then, if $\Phi = (\varphi_{ij})$, $N = (\eta_i\eta_j)$, and $F = (\pi/\Delta)(\varphi_i\varphi_j)$, (3.3) and (3.5) imply

$$\Phi = A\,D^{-1}A^t = I + F - N. \tag{3.8}$$

The diagonal positions here give

$$\eta_i^2 = \frac{\pi}{\Delta}\varphi_i^2 - \varphi_i + 1 \tag{3.9}$$

and from this we record the relation

$$\eta_i^2\eta_j^2 - \frac{\pi^2}{\Delta^2}\varphi_i^2\varphi_j^2 = \frac{\pi}{\Delta}\varphi_i^2(1 - \varphi_j) + (1 - \varphi_i)\,\eta_j^2. \tag{3.10}$$

Now consider the following subsets of $\{1, 2,..., n\}$:

$$\begin{aligned} \mathrm{I} &= \{i \mid \varphi_i = 1,\ \eta_i > 0\}, \\ \mathrm{II} &= \{i \mid \varphi_i = 1,\ \eta_i < 0\}, \\ \mathrm{III} &= \{i \mid \varphi_i < 1\}, \\ \mathrm{IV} &= \{i \mid \varphi_i > 1\}. \end{aligned} \tag{3.11}$$

Since (3.9) excludes $\varphi_i = 1$ and $\eta_i = 0$, we see these classes partition the rows of A. The following lemmas show that, with respect to this partition, the rows of A are "well-behaved":

LEMMA 3.2. If $i, j \in \mathrm{I}$ or if $i, j \in \mathrm{II}$ with $i \neq j$, then $S_i \cap S_j = \varnothing$.

Proof. We have $\varphi_i = \varphi_j = 1$ and $\eta_i\eta_j > 0$. Suppose $S_i \cap S_j \neq \varnothing$. From (3.10) we have

$$\eta_i^2\eta_j^2 = \frac{\pi^2}{\Delta^2}\varphi_i^2\varphi_j^2$$

but $\varphi_{ij} \neq 0$ and

$$\varphi_{ij} + \eta_i\eta_j = \frac{\pi}{\Delta}\varphi_i\varphi_j$$

from (3.7) force

$$\eta_i\eta_j = -\frac{\pi}{\Delta}\varphi_i\varphi_j < 0.$$

LEMMA 3.3. *If $i \in \mathrm{I}$ and $j \in \mathrm{II}$, then $S_i \cap S_j \neq \varnothing$.*

Proof. We have $\varphi_i = \varphi_j = 1$ and $\eta_i\eta_j < 0$. But $\varphi_{ij} = 0$ would give via (3.7)

$$\eta_i\eta_j = \frac{\pi}{\Delta}\varphi_i\varphi_j > 0.$$

LEMMA 3.4. *If $i \in \mathrm{I} \cup \mathrm{II}$ and $j \in \mathrm{III} \cup \mathrm{IV}$, then $S_i \cap S_j \neq \varnothing$.*

Proof. If $S_i \cap S_j = \varnothing$, (3.7) and (3.10) give

$$\frac{\pi}{\Delta}\varphi_i^2(1-\varphi_j) + (1-\varphi_j)\eta_j^2 = 0,$$

but $\varphi_i = 1$ then forces $\varphi_j = 1$.

LEMMA 3.5. *If $i, j \in \mathrm{III}$ or if $i, j \in \mathrm{IV}$, then $S_i \cap S_j \neq \varnothing$.*

Proof. If $S_i \cap S_j = \varnothing$, again we have from (3.7) and (3.10)

$$\frac{\pi}{\Delta}\varphi_i^2(1-\varphi_j) + (1-\varphi_j)\eta_j^2 = 0.$$

Hence $(1-\varphi_i)$ and $(1-\varphi_j)$ have opposite signs.

Now the troublesome case is the one left, namely $i \in \mathrm{III}$, $j \in \mathrm{IV}$. The following lemmas will show that, in such a case, $S_i \cap S_j \neq \varnothing$.

LEMMA 3.6. *Rank* $(\Phi - \mathrm{I}) = 2$.

Proof. The vector $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_n)^t$ lies in the column space of the matrix A, for, if $\boldsymbol{\pi} = (\pi_1, \dots, \pi_{n-1})^t$, we have $\boldsymbol{\varphi} = (1/\pi) A\boldsymbol{\pi}$. Hence $\boldsymbol{\eta}$ and $\boldsymbol{\varphi}$ are orthogonal, so $FN = NF = 0$, and the result is clear from (3.8).

LEMMA 3.7. *Suppose (for notational convenience) $\varphi_1 < 1$ and $\varphi_2 < 1$*

with $\varphi_{12} = 0$. Then $\varphi_{2j} = 0$ only for $j = 1$ and, if $i, j > 2$ and $\varphi_{ij} = 0$, we must have $\varphi_{i1} \neq 0$, $\varphi_{j1} \neq 0$ and $\varphi_i = \varphi_j = 1$.

Proof. Evidently rows one and two of $\Phi - \mathrm{I}$ are independent. Suppose $\varphi_{2j} = 0$ with $j > 2$. Note that $\varphi_{1j} \neq 0$ and look at the submatrix of $\Phi - \mathrm{I}$:

$$\begin{pmatrix} \varphi_1 - 1 & 0 & \varphi_{1j} & \varphi_{1K} \\ 0 & \varphi_2 - 1 & 0 & \varphi_{2K} \\ \varphi_{1j} & 0 & \varphi_3 - 1 & \varphi_{3K} \end{pmatrix}, \tag{3.12}$$

where K is chosen so that $\varphi_{1K} \neq 0$ ($k_1 \geqslant 2$). From the first three columns in (3.12) we deduce that the third row is a negative multiple of the first. But this is inconsistent with $\varphi_{3K} \geqslant 0$. Hence $\varphi_{2j} \neq 0$ for $j \neq 1$. Now suppose $i, j > 2$ and $\varphi_{ij} = 0$ and consider the submatrix

$$(\Phi - \mathrm{I})^{1,2,i}_{1,2,i,j} = \begin{pmatrix} \varphi_1 - 1 & 0 & \varphi_{1i} & \varphi_{1j} \\ 0 & \varphi_2 - 1 & \varphi_{2i} & \varphi_{2j} \\ \varphi_{i1} & \varphi_{i2} & \varphi_i - 1 & 0 \end{pmatrix}. \tag{3.13}$$

If $\varphi_{i1} = 0$, then the third row is a positive multiple of the second, but we know $\varphi_{2j} \neq 0$, thus $\varphi_{ij} \neq 0$. We conclude that $\varphi_{i1} \neq 0$ and similarly that $\varphi_{j1} \neq 0$. Now, since $\varphi_{ij} = 0$, we see from (3.7) and (3.10) that, if $\varphi_i \neq 1$, then $\varphi_j \neq 1$ and in fact $\varphi_i - 1$ and $\varphi_j - 1$ have different signs, say $\varphi_i < 1$ and $\varphi_j > 1$. This means the pair (i, j) satisfy the hypothesis on the pair $(1, 2)$ in the lemma under consideration. We argue that *two* such pairs cannot exist. For convenience we take $i = 3$, $j = 4$ and note that S_2 has a void intersection only with S_1, and S_4 only with S_3. Moreover $S_1 \cap S_3 \neq \varnothing$. Now take S_2 as $\{a_1, ..., a_{r_2}\}$ with $\{a_1\} = S_2 \cap S_3$. If $a_j \in S_1$, the pairs (a_j, a_i), $i = 1, ..., r_2$, determine precisely those sets S_i with $i \geqslant 3$ and $a_j \in S_i$. Since $a_j \notin S_2$, we conclude that $k_j = r_2 + 1$. Analogously, if $a_j \in S_3$ $k_j = r_4 + 1$. But $S_1 \cap S_3 \neq \varnothing$ means then $r_2 = r_4 \equiv r$.

Suppose $a_j \in S_3 - S_1 - S_2$. We have just demonstrated that $k_j = r + 1$ but this cannot be, for a_j belongs to at most $r - 1$ of the sets $S_4, ..., S_n$, as follows: Let $\Gamma_i = \{S_j \mid a_i \in S_j \text{ and } j \geqslant 4\}$, $i = 1, ..., r$. Since S_2 meets every set but S_1, these Γ_i partition $\{S_4, ..., S_n\}$. But our a_j cannot belong to any set of Γ_1, for the set containing both a_j and a_1 is S_3. Further, a_j cannot belong to two S's in the same Γ_i since only one set contains $\{a_j, a_i\}$. We are forced to conclude that $S_3 \subseteq S_1 \cup S_2$ and similarly $S_1 \subseteq S_3 \cup S_4$. This means, of course, that $r_3 = r_1 = 2$. Now we may compute φ_1. There are two ones in row 1 of A corresponding to a_j's which lie in S_1 so these columns have sum $r + 1$ and $\varphi_1 = 2/r$. Just so $\varphi_3 = 2/r$. Also then $\varphi_{13} = \varphi_{23} = 1/r$.

With these values consider the principal submatrix:

$$(3.14)\qquad (\Phi - \mathrm{I})_{1\,2\,3}^{1\,2\,3} = \begin{pmatrix} \dfrac{2-r}{r} & 0 & \dfrac{1}{r} \\ 0 & \varphi_2 - 1 & \dfrac{1}{r} \\ \dfrac{1}{r} & \dfrac{1}{r} & \dfrac{2-r}{r} \end{pmatrix}.$$

Its determinant is given by

$$\frac{1}{r^2}\,(P + r(\varphi_2 - 1)(P^2 - 1)) > 0,$$

where $P = r - 2 \geqslant 1$ since $\varphi_1 < 1$.

This contradiction on the rank of $\Phi - \mathrm{I}$ shows that $\varphi_{ij} = 0$ only if $\varphi_i = \varphi_j = 1$.

We remark here that we have at this point determined the possible structure of AA^t even in the awkward event that some set from class III misses some set from class IV. However, in that event, the structure of AA^t is even simpler than we have thus far determined. In fact, in that case, $\mathrm{I} \cup \mathrm{II} = \varnothing$, as the next lemma shows:

LEMMA 3.8. Let $\mathscr{J} \subseteq \{1,..., n\}$. Suppose $\mathscr{J} \neq \varnothing$ and that the near 1-design $\mathscr{S}$ is such that:

(a) $i, j \in \mathscr{J}, i \neq j \Rightarrow S_i \cap S_j = \varnothing$,

(b) $i \in \mathscr{J}, j \notin \mathscr{J} \Rightarrow S_i \cap S_j \neq \varnothing$.

Then, for $j \notin \mathscr{J}$, S_j cannot miss exactly one S_i.

Proof. Take $\mathscr{J} = \{2,..., e + 1\}$ and, choosing row 1 arbitrarily (not in $\mathscr{J}$), write the matrix A in the form

$$(3.15)\qquad \begin{array}{c|c|c}
\overbrace{1 \cdots 1\ \ 1 \cdots 1}^{r_1} & 0 \cdots 0 & 0 \cdots 0 \\
\begin{matrix} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \end{matrix} & \begin{matrix} 1\text{-}1 & & \\ & \ddots & \\ & & 1\text{-}1 \end{matrix} & 0 \\ \hline
\begin{matrix} \vec{1} & & & \\ & \vec{1} & & \\ & & \ddots & \\ & & & \vec{1} \end{matrix} & \begin{matrix} A_{1,1} \cdots A_{1,e} \\ A_{2,1} \cdots A_{2,e} \\ \vdots \\ A_{r_1,1} \cdots A_{r_1,e} \end{matrix} & * \\ \hline
0 & C & *
\end{array}$$

Recall we may assume $k_j > 2$ and that $\mathbf{1}$ denotes a string of ones.

Now evidently $1 \leqslant e \leqslant r_1$ and one easily sees that $A_{ii} = 0$ and A_{ij} for $i \neq j$ is a permutation matrix. These observations yield the relations

$$(3.16)\qquad \begin{aligned} k_i &= r_{j+1} + 1, \qquad 1 \leqslant i \leqslant e,\ 1 \leqslant j \leqslant e;\ i \neq j; \\ k_i &= r_{j+1}, \qquad e + 1 \leqslant i \leqslant r_1,\ 1 \leqslant j \leqslant e;\ i \neq j. \end{aligned}$$

Unless $e = 2$, this gives $r_2 = \cdots = r_{e+1}$. If $e = 2$ but $r_1 \geqslant 3$, A_{32} square forces $r_2 = r_3$. We have thus established that $r_2 = \cdots = r_{e+1}$ unless $e = 2$ and $i \notin \mathscr{J}$ implies $r_i = 2$. But this latter is not possible since the totality of ones outside rows two and three is then

$$2(n-2) = \sum_{j=1}^{r_2+r_3} (k_j - 1) + \sum_{j>r_2+r_3} k_j \geqslant \sum_{j=1}^{n-1} (k_j - 1) \geqslant 2(n-1).$$

Hence $r_2 = r_3 = \cdots = r_{e+1} \equiv r$. Then (3.16) says

$$(3.17)\qquad \begin{aligned} k_1 &= \cdots = k_e = r + 1, \\ k_{e+1} &= \cdots = k_{r_1} = r. \end{aligned}$$

Now each a_j necessarily belongs to some S_i with $i \notin \mathscr{J}$ so we have proved that

$$a_j \in \bigcup_{i \in \mathscr{J}} S_i \Rightarrow k_j = r + 1 \qquad \text{and} \qquad a_j \notin \bigcup_{i \in \mathscr{J}} S_i \Rightarrow k_j = r.$$

Now in (3.15) suppose row one misses only row n. Then C is a row vector and evidently all column through its components have sum $r + 1$ since there are ones in these column in the rows of $\mathscr{J}$. But the column sums above C are seen to be constant, r_1, so that C is a vector of ones and $e \leqslant 2$, $r = r_1 = 2$, $n \leqslant 4$.

LEMMA 3.9. *If $i \in \mathrm{III}$ and $j \in \mathrm{IV}$, then $S_i \cap S_j \neq \varnothing$.*

Proof. If $\varphi_{ij} = 0$, we know that S_j misses only S_i and in view of Lemma 3.8 we must conclude that $\mathrm{I} = \mathrm{II} = \varnothing$. Now the only zero entries in the matrix $\Phi - I$ occur in row i and column i. This means that if we delete row i from A, we obtain a square matrix with row inner products one. Such a matrix has column inner products one, implying that the deleted row had at most one non-zero entry, contrary to $r_i \geqslant 2$.

We summarize all of the preceding in

LEMMA 3.10. *Let A represent a near 1-design. Then we may permute the rows of A obtaining A_1 satisfying*

$$A_1 A_1^t = \begin{bmatrix} B_1 & J & J \\ J & B_2 & J \\ J & J & B_3 \end{bmatrix}, \tag{3.18}$$

where B_1 and B_2 are diagonal matrices and the off-diagonal entries of B_3 are one. (Nothing is asserted about the size of B_i.)

Proof. Place the rows of class I first, then those of class II, and finally those of III $\cup$ IV. Then (3.18) is a consequence of Lemmas 3.2, 3.3, 3.4, 3.5, and 3.9.

We now prove Theorem 2.1. Let A carry a near 1-design normalized as in (3.18). Since the matrix (3.18) is singular, not both of B_1 and B_2 can be 1×1. Now suppose exactly *one* of the classes I or II contains at least two elements. Adjoin to A a column with ones in the positions of this class and zeros elsewhere. The resulting matrix is evidently a projective plane (note there is a row with sum at least three) so that A is a partial plane.

The remaining case is that in which neither I nor II contains fewer than 2 elements. As above we may evidently obtain a matrix C with row inner products one, this time by adjoining *two* columns to the matrix A, one for class I and the other for class II. C will be $n \times (n + 1)$ and $E = C^t$ carries a near 1-design. Apply Lemma 3.10 to this matrix E and note that the 2 columns added to A correspond to 2 rows of E with zero inner product. If these two rows meet, all the other rows of E are in the case above whereby adjoining a column with two ones produces a projective plane. This, of course, cannot be, for E has row sums of three or more. It must be that these two rows which miss each other miss exactly *one* other row. They cannot miss more than one for, according to (3.18), those rows would miss each other and we are dealing with columns of the original matrix A. Thus, in (3.18), we see we have one B_i of order 3, the other of order less than 2, so that by adjoining now a column to E containing three ones we obtain a projective plane, evidently of order 2. This is precisely the reverse procedure for constructing the design (2.1)–(2.2).

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